Phys 410
Spring 2013
Lecture #37 Summary
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We considered the most general coupled oscillator problem – N particles coupled to each other by means of springs or any other types of forces that produce a stable equilibrium configuration. This system has n generalized coordinates, where in general $n \neq N$. The generalized coordinates are written as $\vec{q} = (q_1, q_2, \dots q_n)$. We assume that only conservative forces act between the particles, hence (as known from previous studies) the potential energy is a function only of the coordinates: $U = U(\vec{q})$. The kinetic energy is that of all of the particles in the system: $T = \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha^2$. The "raw" coordinates $\vec{r}_\alpha$can be written in terms of the generalized coordinates as $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \dots q_n)$, where it is assumed that no explicit time-dependence is required to write down this transformation. The kinetic energy can be written as $T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\, \dot{q}_i \dot{q}_j$, where the matrix $\bar{\bar{A}}$ is defined as $A_{ij} \equiv \sum_{\alpha=1}^{N} m_\alpha \frac{\partial \vec{r}_\alpha}{\partial q_i}\frac{\partial \vec{r}_\alpha}{\partial q_j}$. Note that the double pendulum kinetic energy (see the Lagrangian above) has a kinetic energy of this form, including a $\dot{q}_1\dot{q}_2$ term. Note that the matrix $\bar{\bar{A}}$ is a function of the generalized coordinates as well: $\bar{\bar{A}} = \bar{\bar{A}}(\vec{q})$. We now have the full Lagrangian of this generalized coupled oscillator problem $\mathcal{L} = T\left(\vec{q}, \dot{\vec{q}}\right) - U(\vec{q})$.

We next considered the small oscillation motion of the system around a stable equilibrium point. This means that we will keep terms only up to second order in the variables. By a shift of the origin, we can make the stable equilibrium point appear at the point $\vec{q} = (0, 0, \dots 0)$. We then did a Taylor series expansion of the potential around this point and kept terms up to second order, yielding $U(\vec{q}) = \frac{1}{2}\sum_{i,j} K_{ij} q_i q_j$, where the matrix elements of $\bar{\bar{K}}$ are the curvatures of the potential with respect to the generalized coordinates: $K_{ij} \equiv \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\vec{q}=0}$. The kinetic energy is already quadratic in the variables, so we simply evaluate it at $\vec{q} = 0$ to yield $T = \frac{1}{2}\sum_{i,j} A_{ij}(0)\dot{q}_i\dot{q}_j \quad = \frac{1}{2}\sum_{i,j} M_{ij}\dot{q}_i\dot{q}_j$, where the mass matrix $\bar{\bar{M}}$ is the $\bar{\bar{A}}$ matrix evaluated at the equilibrium position $\vec{q} = (0, 0, \dots 0)$. The Lagrangian $\mathcal{L} = T\left(\dot{\vec{q}}\right) - U(\vec{q})$ is now a homogeneous quadratic function of the coordinates and their time-derivatives, and the matrices $\bar{\bar{M}}$ and $\bar{\bar{K}}$ are constant symmetric real matrices.

There are n Lagrange equations to set up and solve. We wrote down the equations and found that the set of n equations are summarized beautifully in a simple matrix equation: $-\bar{\bar{K}}\vec{q} = \bar{\bar{M}}\ddot{\vec{q}}$. We can solve this equation using the same method employed before, just generalized to n coordinates. We use the complex *ansatz* for the solution vector: $\vec{q}(t) =$

$Re\left[\vec{C}e^{i\omega t}\right]$, where $\vec{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}$, and the C_i are complex constants. Putting this into the matrix equation yields $\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right)\vec{C} = 0$. To get a non-trivial solution for $\vec{C}$, we demand that $det\left(\bar{\bar{K}} - \omega^2\bar{\bar{M}}\right) = 0$. This yields an n-th order equation for ω^2, with n real solutions (we know this because the matrix $\bar{\bar{K}} - \omega^2\bar{\bar{M}}$ is real and symmetric). The n normal modes follow by standard linear algebra.